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On brane actions and superembeddings

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Abstract: Actions for branes, with or without worldsurface gauge fields, are discussed in a unified framework. A simple algorithm is given for constructing the component Green-Schwarz actions. Superspace actions are also discussed. Three examples are given to illustrate the general procedure: the membrane in $D = 11$ and the D2-brane, which both have on-shell worldsurface supermultiplets, and the membrane in $D = 4$, which has an off-shell multiplet.

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Contents

1. Introduction

In the superembedding approach to supersymmetric extended objects the object under consideration is described mathematically as a subsupermanifold (the worldsurface) of superspacetime (the target supermanifold). This approach was initiated some time ago [1, 2] in the context of superparticles. Superspace actions were found for particles in $D = 3, 4, 6$ and 10 dimensional spacetimes [1, 2, 3, 4, 5, 6] and then later for the heterotic string in ten dimensions [7, 8]. Actions of the heterotic string type were constructed for other type I branes (i.e. branes with no worldsurface vector or tensor fields) [9, 10] but it was not clear at the time that these actions described the right degrees of freedom. In [11] a generalised action was proposed for type I branes which leads to the standard Green-Schwarz equations of motion (see [12, 13] for reviews) and this approach has recently been extended to cover D-branes [14, 15, 16].

The structure of the worldsurface supermultiplets that arise in the superembedding formalism was clarified in [17]; there it was assumed that a natural embedding condition, namely that the odd tangent bundle of the worldsurface should be a subbundle of the pull-back of the odd tangent bundle of the target space, holds. It was found in [17] that three types of multiplet can arise: on-shell, off-shell or underconstrained. In the on-shell case, there can be no superspace actions of the heterotic string type since such actions would necessarily involve the propagation of the Lagrange multipliers that are used in this construction. Nevertheless, on-shell embeddings are useful for deriving equations of motion; for example, the full equations of motion of the M-theory five-brane were first obtained this way [18]. In the off-shell case, by which it is meant that the wordsurface multiplet is a recognisable off-shell multiplet, it is possible to write down actions of the heterotic string type. The third case that arises, and which we call underconstrained here, typically occurs for branes with low codimension. For example, in codimension one the basic embedding condition gives rise to an unconstrained scalar superfield. In order to get a recognisable multiplet further constraints must be imposed. An example of this is given by IIA D-branes where the basic embedding condition yields an on-shell multiplet for $p = 0, 2, 4$, but an underconstrained one for $p = 6, 8$. By imposing by hand the further constraint that there is a worldsurface vector field with the usual modified Bianchi identity whose superspace field strength vanishes unless all indices are bosonic one recovers on-shell multiplets [19]. (For $p = 0, 2, 4$ one can show that the vector Bianchi identity follows from the basic embedding condition.)

In this note we show that there is a simple algorithm for generating actions for (almost) all branes starting from the superembedding formalism. It can be used in two ways: if the multiplet is on-shell, one can use it to find the Green-Schwarz action; if the multiplet is off-shell one can use it either to write down a superspace action of heterotic string type or one can construct a Green-Schwarz action which in general will have auxiliary fields. In the underconstrained case we shall assume that further constraints have been imposed to convert the embedding into one of the first two types. The actions obtained this way are Lorentz covariant and are thus not applicable to branes with self-dual tensor multiplets, although actions involving additional fields have been proposed for these cases [20, 21]. We give three examples: the $D = 11$ supermembrane, which has an on-shell scalar multiplet, the type IIA $D2$-brane in $D = 10$, which has an on-shell vector multiplet, and the membrane in $D = 4$, which is off-shell.

The method of constructing actions proposed here is closely related both to the superspace method used for the heterotic string and to the generalised action principle. However, the proof that the GS action is κ-symmetric is greatly simplified. In addition, our approach is deductive in the sense that we derive the GS action from the superembedding formalism. Thus, in the case of D-branes, rather than starting with the Dirac-Born-Infeld (DBI) term in the action we show that it emerges from the construction. An advantage of this approach is that it is applicable to other type II branes which have higher rank worldsurface antisymmetric tensor gauge fields, provided that they are not self-dual.

2. Superembeddings

We consider superembeddings $f : M \to \underline{M}$, where the worldsurface M has (even|odd) dimension $(d|\frac{1}{2}D')$ and the target space has dimension $(D|D')$. In local coordinates M is given as $z^{\underline{M}}(z^M)$, where $z^{\underline{M}} = (x^{\underline{m}}, \theta^{\underline{m}})$ and $z^M = (x^m, \theta^\mu)$ (if no indices are used we shall distinguish target space coordinates from worldsurface ones by underlining the former). The embedding matrix $E_A{}^{\underline{A}}$ is defined to be

$$E_A{}^{\underline{A}} = E_A{}^M \partial_M z^{\underline{M}} E_{\underline{M}}{}^{\underline{A}} \ , \tag{2.1}$$

in other words, the embedding matrix is the differential of the embedding map referred to standard bases on both spaces. Our index conventions are as follows: latin (greek) indices are even (odd) while capital indices run over both types; letters from the beginning of the alphabet are used to refer to a preferred basis while letters from the middle of the alphabet refer to a coordinate basis, the two types of basis being related to each other by means of the vielbein matrix $E_M{}^A$ and its inverse $E_A{}^M$; exactly the same conventions are used for the target space and the worldsurface with the difference that the target space indices are underlined. Primed indices are used to denote directions normal to the worldsurface. We shall also use a two-step notation for worldsurface spinor indices where appropriate: in general discussions, a worldsurface spinor index such as α runs from 1 to $\frac{1}{2}D'$, but it may often be the case that the group acting on this index includes an internal factor as well as the spin group of the worldsurface; in this case we replace the single index α with the pair αi where i refers to the internal symmetry group. A similar convention is used for normal spinor indices.

The basic embedding condition is

$$E_\alpha{}^{\underline{a}} = 0 \ . \tag{2.2}$$

It implies that the odd tangent space of the worldsurface is a subspace of the odd tangent space to $\underline{M}$ at each point in $M \subset \underline{M}$. In many cases, equation (2.2) determines the equations of motion for the brane under consideration. Moreover, it also determines the geometry induced on the worldsurface and implies constraints on the background geometry which arise as integrability conditions for the existence of such superembeddings. For the cases where the worldsurface multiplet is underconstrained one can arrive at a multiplet which describes the physical fields by imposing the further constraint that there should exist appropriate q-form worldsurface gauge fields, $\mathcal{F}_q$. We will describe this constraint in the case of Dp-branes below.

In addition to the embedding matrix, each brane comes with a Wess-Zumino form, W_{p+2}, defined on M. This term takes different forms for different branes. To be specific, let us consider the fundamental Fp-branes and Lp-branes [17] with 16 target space supersymmetries and Dp-branes and the M5-branes which have 32 target space supersymmetries. The F-class corresponds to p-branes in $p+5$ dimensions ($p = 1, 2..., 5$) and the L-class corresponds to p-branes in $p+4$ dimensions ($p = 1, 2, ..., 5$). In each one of these cases there exist Cartan integrable systems in the target space which take the form

$$\begin{aligned}
Fp \ &: \quad dG_{p+2} = 0 \ , \\
Lp \ &: \quad dG_{p+2} = G_2 G_{p+1} \ , \qquad dG_{p+1} = 0 \ , \qquad dG_2 = 0 \ , \\
Dp \ &: \quad dG = GH_3 \ , \qquad dH_3 = 0 \ , \\
M5 \ &: \quad dG_7 = G_4 G_4 \ , \qquad dG_4 = 0 \ ,
\end{aligned} \tag{2.3}$$

where, in the Dp-brane case, G is a sum of the Ramond-Ramond (RR) curvatures which have even/odd ranks in type IIA/B theory, and wedge products of forms are understood. These equations can be solved locally to give

$$
\begin{array}{llll}
Fp & : & G_{p+2} = dC_{p+1} \ , & \\
Lp & : & G_{p+2} = dC_{p+1} - C_1 G_{p+1} \ , & G_{p+1} = dC_p \ , \ G_2 = dC_1 \ , \\
Dp & : & G = dC - CH_3 + me^{B_2} \ , & H_3 = dB_2 \ , \\
M5 & : & G_7 = dC_6 - C_3 G_4 \ , & G_4 = dC_3 \ ,
\end{array}
\tag{2.4}
$$

where m is an arbitrary constant which is relevant for type IIA theory and C is the sum of the RR potentials. We denote by C the potentials associated with all the target space field strengths, with the exception of $H_3 = dB_2$ which plays a special role in the case of Dp-branes.

The Wess-Zumino form W_{p+2} is a closed form

$$dW_{p+2} = 0 \ , \tag{2.5}$$

constructed from from the pull-backs of suitable target space forms as well as intrinsic worldvolume forms. For Fp-branes the form G_{p+2} is closed, and therefore its pullback to the worldvolume is a candidate Wess-Zumino form. However, the forms G_{p+2} in Dp-brane case, G_7 in the M5-brane case and G_{p+2} in the L_p-brane case are not closed. This is remedied by introducing respectively a two-form $\mathcal{F}_2$, a three-form $\mathcal{F}_3$ and a p-form $\mathcal{F}_p$ as follows:

$$Lp \quad : \quad \mathcal{F}_p = dA_{p-1} - f^* C_p \ , \tag{2.6}$$

$$Dp \quad : \quad \mathcal{F}_2 = dA_1 - f^* B_2 \ , \tag{2.7}$$

$$M5 \quad : \quad \mathcal{F}_3 = dA_2 - f^* C_3 \ , \tag{2.8}$$

These satisfy the Bianchi identities

$$Lp \quad : \quad d\mathcal{F}_p = -f^* G_{p+1} \ , \tag{2.9}$$

$$Dp \quad : \quad d\mathcal{F}_2 = -f^* H_3 \ , \tag{2.10}$$

$$M5 \quad : \quad d\mathcal{F}_3 = -f^* G_4 \ . \tag{2.11}$$

Note that the construction of these forms has led to the introduction of intrinsic worlvolume potentials A_1, A_2 and A_{p-1}. Using the ingredients described above, we construct the Wess-Zumino forms as follows:

$$
W_{p+2} = \begin{cases}
f^* G_{p+2} & \mathrm{F}p \\
f^* \left(G_{p+2} \ + \mathcal{F}_p G_2 \right) & \mathrm{L}p \\
\left((f^* G) e^{\mathcal{F}} \right)_{p+2} & \mathrm{D}p \\
f^* \left(G_7 + \mathcal{F}_3 G_4 \right) & \mathrm{M5}
\end{cases}
\tag{2.12}
$$

It is easy to verify that all these forms are indeed closed. Thus, the Wess-Zumino form W_{p+2} can *locally* be written as

$$W_{p+2} = dZ_{p+1} \ , \tag{2.13}$$

where

$$Z_{p+1} = \begin{cases} f^*C_{p+1} & \mathrm{F}p \\ f^*\left(C_{p+1} + C_1\mathcal{F}_p\right) & \mathrm{L}p \\ \left((f^*C)e^{\mathcal{F}}\right)_{p+1} + m\omega_{p+1} & \mathrm{D}p \\ f^*\left(C_6 + C_3\mathcal{F}_3\right) & \mathrm{M5} \end{cases} \tag{2.14}$$

and where $\omega_{p+1}(A, dA)$ is the Chern-Simons form present for type IIA Dp-branes defined by

$$d\omega_{p+1}(A, dA) = (e^{dA})_{p+2} \ . \tag{2.15}$$

We mentioned earlier that for the cases where the worldsurface multiplet is under-constrained one can arrive at a multiplet which describes the physical fields by imposing the further constraint on a suitable worldvolume superform. In the case of Dp-branes that constraint is [22]

$$\mathcal{F}_{\alpha B} = 0 \ , \tag{2.16}$$

i.e. all of the components of $\mathcal{F}$ except the purely bosonic ones must vanish. It can be shown that for $p < 6$ the basic embedding condition puts the theory on-shell [19], and that for these cases the condition (2.16) follows automatically. In other cases one can argue for these constraints by considering open branes which end on other branes [22, 23]. A similar situation arises for the M5-brane, for which we refer the reader to refs. [18, 24]. The case of Lp-branes will be treated in detail elsewhere [25].

3. Kappa symmetry and Green-Schwarz actions

The basic embedding condition (2.2), which underlies all branes studied so far and which is geometrically very natural, is intimately related to κ-symmetry in the GS approach to branes. Under an infinitesimal worldsurface diffeomorphism one has

$$(\delta z^{\underline{M}})E_{\underline{M}}{}^{\underline{A}} = v^A E_A{}^{\underline{A}} \ , \tag{3.1}$$

where v^A is the worldsurface vector field generating the diffeomorphism. For an odd diffeomorphism, with $v^a = 0$, one finds, using the embedding condition (2.2),

$$\delta z^{\underline{a}} \equiv (\delta z^{\underline{M}})E_{\underline{M}}{}^{\underline{a}} = 0 \ , \tag{3.2}$$

and

$$\delta z^{\underline{\alpha}} \equiv (\delta z^{\underline{M}})E_{\underline{M}}{}^{\underline{\alpha}} = v^\alpha E_\alpha{}^{\underline{\alpha}} \ . \tag{3.3}$$

This can be rewritten in the more usual κ-symmetry form

$$\delta z^{\underline{\alpha}} = \frac{1}{2}\kappa^{\underline{\beta}}(1+\Gamma)_{\underline{\beta}}{}^{\underline{\alpha}} \,, \tag{3.4}$$

where

$$\kappa^{\underline{\alpha}} = v^{\alpha} E_{\alpha}{}^{\underline{\alpha}} \tag{3.5}$$

and where

$$P_{\underline{\alpha}}{}^{\underline{\beta}} = \frac{1}{2}(1+\Gamma)_{\underline{\alpha}}{}^{\underline{\beta}} \tag{3.6}$$

is the projection operator onto the odd tangent space of the worldsurface from the odd tangent space of the target. It is given in terms of $E_{\alpha}{}^{\underline{\alpha}}$ by

$$P_{\underline{\alpha}}{}^{\underline{\beta}} = (E^{-1})_{\underline{\alpha}}{}^{\gamma} E_{\gamma}{}^{\underline{\beta}} \,. \tag{3.7}$$

Thus we have

$$\delta z^{\underline{a}} = 0 \tag{3.8}$$

$$\delta z^{\underline{a}} = \frac{1}{2}\kappa^{\underline{\beta}}(1+\Gamma)_{\underline{\beta}}{}^{\underline{\alpha}} \,. \tag{3.9}$$

Equations (3.9), evaluated at $\theta = 0$, are the standard κ-symmetry transformations of $z^{\underline{M}}(x)$ in the GS formalism. The explicit form of the operator Γ, which must square to unity in order for P to be a projector, and the explicit relation between the parameters for κ-symmetry and worldsurface supersymmetry depend on the choice of basis for the odd tangent space on the worldsurface, but whichever basis one chooses to work with, κ-symmetry will have a precise definition in terms of worldsurface supersymmetry. Of course the latter does not change and so should, we would argue, be thought of as being more fundamental.

For any brane the Wess-Zumino form W_{p+2} is closed. Since it is a $p+2$-form on a manifold which has even (i.e. bosonic) dimension $p+1$ it follows that it is exact. This is so because the de Rham cohomology of a supermanifold coincides with the de Rham cohomology of its body. Therefore we can always write

$$W_{p+2} = dK_{p+1} \tag{3.10}$$

for some *globally* defined $(p+1)$-form K on M. Furthermore, since none of the target space fields or the worldsurface fields has negative dimension, at least for the models under discussion here, it follows that the only non-vanishing component of K is the purely bosonic one. In components this means

$$K_{\alpha A_1 \cdots A_p} = 0 \,. \tag{3.11}$$

We now define the Green-Schwarz Lagrangian form L_{p+1} to be

$$L_{p+1} = K_{p+1} - Z_{p+1} \tag{3.12}$$

Under a worldsurface diffeomorphism generated by the vector field v one has

$$\delta L_{p+1} = \mathcal{L}_v L_{p+1} = d i_v L_{p+1} + i_v d L_{p+1} \,. \tag{3.13}$$

Since, by construction, L_{p+1} is closed,

$$dL_{p+1} = 0 \,, \tag{3.14}$$

the variation (3.13) reduces to

$$\delta L_{p+1} = d i_v L_{p+1} \,. \tag{3.15}$$

Therefore the action integral

$$S = \int_{M_0} L^0_{p+1} \,, \tag{3.16}$$

where M_0 is the body of M and where

$$L^0_{p+1} = dx^{m_{p+1}} \wedge dx^{m_p} \wedge \ldots dx^{m_1} L_{m_1 \ldots m_{p+1}}| \,, \tag{3.17}$$

where the vertical bars indicate evaluation of a (worldvolume) superfield at $\theta = 0$, will be invariant under κ-symmetry transformations and diffeomorphisms of M_0, since these transformations are identified with the leading components of the superdiffeomorphisms of M.

As we noted in the introduction, this result is closely related to both superspace actions of the heterotic string type and to the generalised actions of refs [11, 14, 15]. However, there is a difference in that, in the generalised action formalism [11, 14, 15], the Dirac-Born-Infeld action is explicitly included in the case of D-branes. The method proposed here generates the DBI action (from K_{p+1}) automatically, and moreover allows for the DBI action to be extended to worldsurface q-form gauge fields with $q > 2$. The argument given above shows that the Lagrangian we have constructed is invariant uder the right symmetries and has the usual Wess-Zumino term. The contribution to the action from K must therefore be the DBI action. Below we shall show that this is indeed the case in specific examples.

It is worth emphasizing that not only is the DBI action automatically generated in the method proposed here, but that also the κ-symmetry of the total action is made manifest. This is due to the closure property (3.15). In the generalized action formalism, however, while the action is indeed an integral of a Lagrangian $(p+1)$-form over M_0, not only is the DBI term explicitly included (along with certain Lagrange multiplier terms), but also the closure property $dL_{p+1} = 0$, needed for the proof of k-symmetry, is non-manifest, and proving it requires lengthy calculations [14, 15].

The form of the Lagrangian given in (3.12) is closely related to the actions considered before in [7] for the heterotic string, in [9] for the $D = 11$ supermembrane, and in [10] for higher super p-branes. We shall comment about this relation in more detail in Section 7.

4. M2-brane

To illustrate the above general formalism we consider first the simplest case, namely an on-shell type I brane, the membrane (M2-brane) in $D = 11$. We assume that the embedding condition (2.2) holds. (See [26] as well for a treatment of the supermembrane in the superembedding formalism). It can then be shown that we may choose

$$E_\alpha{}^{\underline{\alpha}} = u_\alpha{}^{\underline{\alpha}}, \tag{4.1}$$

$$E_a{}^{\underline{a}} = u_a{}^{\underline{a}}, \tag{4.2}$$

with the complementary normal matrix $E_{A'}{}^{\underline{A}}$, which specifies the choice of normal spaces, being given by

$$E_{\alpha'}{}^{\underline{\alpha}} = u_{\alpha'}{}^{\underline{\alpha}}, \tag{4.3}$$

$$E_{a'}{}^{\underline{a}} = u_a{}^{\underline{a}}. \tag{4.4}$$

We may also impose

$$E_{\alpha'}{}^{\underline{a}} = 0 \ . \tag{4.5}$$

In these formulae u denotes an element of the group $Spin(1,10)$ or the corresponding element of the Lorentz group in eleven dimensions. Thus the matrices $u_\alpha{}^{\underline{\alpha}}$ and $u_{\alpha'}{}^{\underline{\alpha}}$ together make up an element of $Spin(1,10)$ while $u_a{}^{\underline{a}}$ and $u_{a'}{}^{\underline{a}}$ make up the corresponding element of $SO(1,10)$. We remind the reader that although $E_\alpha{}^{\underline{a}} = 0$, it is not the case that $E_a{}^{\underline{\alpha}} = 0$, although we can choose

$$E_a{}^{\underline{\alpha}} = \Lambda_a{}^{\alpha'} u_{\alpha'}{}^{\underline{\alpha}} \ . \tag{4.6}$$

The leading component of the superfield $\Lambda_a{}^{\alpha'}$ should be thought of as the spacetime derivative of the transverse fermionic coordinate field, that is, the derivative of the physical fermion field of the membrane.

In order to derive the GS action from the superembedding formalism it is necessary to show that the $\theta = 0$ component of $E_m{}^a$, which we denote by $\mathcal{E}_m{}^a$ is the dreibein for the GS metric. The latter is defined to be

$$g_{mn} = \left(\partial_m z^{\underline{M}} E_{\underline{M}}{}^{\underline{a}}\right)\left(\partial_n z^{\underline{N}} E_{\underline{N}}{}^{\underline{b}} \eta_{\underline{ab}}\right)| \ , \tag{4.7}$$

where, we recall that the bar denotes evaluation of a quantity at $\theta = 0$. From the embedding condition we have

$$E_\alpha{}^{\underline{a}} = E_\alpha{}^m \left(\partial_m z^{\underline{M}}\right) E_{\underline{M}}{}^{\underline{a}} + E_\alpha{}^\mu \left(\partial_\mu z^{\underline{M}}\right) E_{\underline{M}}{}^{\underline{a}} = 0 \ . \tag{4.8}$$

We can always choose a gauge on the worldvolume such that $E_\alpha{}^m| = 0$. Moreover the leading component of $E_\alpha{}^\mu$ is non-singular. Therefore, evaluating the above equation at $\theta = 0$ we deduce

$$\partial_\mu z^{\underline{M}} E_{\underline{M}}{}^{\underline{a}}| = 0 \ . \tag{4.9}$$

JHEP08(1998)011

Using this result we find

$$E_a{}^{\underline{a}}| = E_a{}^m \left(\partial_m z^{\underline{M}}\right) E_{\underline{M}}{}^{\underline{a}}| \ . \tag{4.10}$$

It then follows, since $E_a{}^m| = \mathcal{E}_a{}^m$, the inverse of $\mathcal{E}_m{}^a$, and the fact that

$$E_a{}^{\underline{a}} E_b{}^{\underline{b}} \eta_{\underline{ab}} = \eta_{ab} \ , \tag{4.11}$$

that $\mathcal{E}_m{}^a$ is indeed the dreibein for the GS metric as claimed, i.e.

$$\mathcal{E}_m{}^a \mathcal{E}_n{}^b \eta_{ab} = g_{mn} \ . \tag{4.12}$$

The Wess-Zumino form for the M2-brane is the pull-back of the supergavity four-form G_4. Its non-vanishing components are

$$G_{\underline{\alpha\beta cd}} = -i(\Gamma_{\underline{cd}})_{\underline{\alpha\beta}} \tag{4.13}$$

and the totally vectorial component $G_{\underline{abcd}}$. On the worldvolume of the brane there should therefore be a three-form K_3 such that

$$W_4 = f^* G_4 = dK_3 \ . \tag{4.14}$$

In index notation this reads

$$4\nabla_{[A} K_{BCD]} + 6T_{[AB}{}^E K_{|E|CD]} = (f^* G)_{ABCD} \ . \tag{4.15}$$

This is indeed the case as we shall now verify. Since there are no fields of negative dimension on the worldvolume (given the standard embedding condition), the only non-vanishing component of K has purely vectorial indices. By directly evaluating the dimension zero component of the above equation one finds that it is satisfied for

$$K_{abc} = \epsilon_{abc} \ . \tag{4.16}$$

Since there are no fields of negative dimension it is apparent that the negative dimension compoents of $W_4 = dK_3$ are trivially satisfied. To prove that the remaining components are also satisfied it is convenient to introduce a four-form I_4 defined by

$$I_4 = W_4 - dK_3 \ , \tag{4.17}$$

where K_3 has the components described above. Clearly $dI_4 = 0$. We need to show that $I_4 = 0$ but, by dimensional analysis, the only components of I_4 that need to be checked are $I_{\alpha\beta cd}$ and $I_{\alpha bcd}$ (since I_{abcd} vanishes identically). The fact that $I_{\alpha\beta cd}$ vanishes can easily be checked using the formulae given above while one can show that this implies automatically that $I_{\alpha bcd} = 0$ by using the identity $dI_4 = 0$. In a coordinate basis one therefore has

$$K_{mnp}| = \epsilon_{mnp}\sqrt{-\det g} \ , \tag{4.18}$$

where g is the GS metric. The GS Lagrangian is therefore recovered from the general formulae (3.12) and (3.16); it is

$$\mathcal{L} = \sqrt{-\det g} - \frac{1}{6}\epsilon^{mnp}\partial_p z^{\underline{P}}\, \partial_n z^{\underline{N}}\, \partial_m z^{\underline{M}}\, C_{\underline{MNP}} \ , \tag{4.19}$$

where $G_4 = dC_3$ on $\underline{M}$, and where

$$L^0 = dx^m \wedge dx^n \wedge dx^p \epsilon_{mnp}\ \mathcal{L} \ . \tag{4.20}$$

5. D2-brane

The on-shell example we shall consider is the IIA $D2$-brane in $D=10$. For simplicity we shall take the target space to be flat and $m=0$, although this is not essential. The basic embedding equation (2.2) is imposed as usual and we may choose to parametrise the dimension zero components of the embedding matrix in the form [18]

$$\begin{aligned} E_\alpha{}^{\underline{\alpha}} &= u_\alpha{}^{\underline{\alpha}} + h_\alpha{}^{\beta'} u_{\beta'}{}^{\underline{\alpha}} , \\ E_a{}^{\underline{a}} &= u_a{}^{\underline{a}} . \end{aligned} \tag{5.1}$$

Here u denotes part of a matrix of the group $Spin(1,9)$, in the spinor or the vector representations according to the indices. The pull-back to the worldsurface of the defining equation for the target space torsion two-form gives the equation

$$\nabla_A E_B{}^{\underline{C}} - (-1)^{AB} \nabla_B E_A{}^{\underline{C}} + T_{AB}{}^C E_C{}^{\underline{C}} = (-1)^{A(B+\underline{B})} E_B{}^{\underline{B}} E_A{}^{\underline{A}} T_{\underline{AB}}{}^{\underline{C}} . \tag{5.2}$$

The dimension zero component of this equation reads, on using the embedding condition (2.2),

$$E_\alpha{}^{\underline{\alpha}} E_\beta{}^{\underline{\beta}} T_{\underline{\alpha\beta}}{}^{\underline{c}} = T_{\alpha\beta}{}^c E_c{}^{\underline{c}} . \tag{5.3}$$

Using

$$T_{\underline{\alpha\beta}}{}^{\underline{c}} = -i(\Gamma^{\underline{c}})_{\underline{\alpha\beta}} \tag{5.4}$$

and (5.1) one finds [19]

$$h_\alpha{}^{\beta'} \to h_{\alpha i}{}^{\beta' j} = \delta_i{}^j (\gamma^{ab})_\alpha{}^{\beta'} h_{ab} \tag{5.5}$$

and

$$T_{\alpha\beta}{}^c = -i(\Gamma^d)_{\alpha\beta} m_d{}^c , \tag{5.6}$$

where

$$m_a{}^b = \delta_a{}^b (1-4y) + 8(h^2)_a{}^b \tag{5.7}$$

with

$$y = \frac{1}{2} \mathrm{tr}\, h^2 \tag{5.8}$$

and where h^2 denotes matrix multiplication.

It is not difficult to show that the embedding condition (2.2) implies the existence of a two-form $\mathcal{F}$ such that

$$d\mathcal{F} = -f^* H_3 , \tag{5.9}$$

where H_3 is the pull-back of the NS three-form on the target space. This identity is satisfied provided that we choose all the components of $\mathcal{F}$ to vanish except for F_{ab} which is related to h by

$$m_a{}^c \mathcal{F}_{cb} = 4h_{ab} . \tag{5.10}$$

This can be rearranged to give

$$\mathcal{F}_{ab} = \frac{4h_{ab}}{1+4y} . \tag{5.11}$$

The Wess-Zumino four-form W_4 is given by

$$W_4 = dZ_3 = d(f^*C_3 + f^*C_1\mathcal{F}) \ , \tag{5.12}$$

where C_1 and C_2 are two of the RR potentials on the target space. It can be rewritten as

$$W_4 = f^*G_4 + f^*G_2\mathcal{F} \ , \tag{5.13}$$

where the RR field strengths G_4 and G_2 are given by

$$G_4 = dC_3 + B_2G_2 \,, \tag{5.14}$$

$$G_2 = dC_1 \,, \tag{5.15}$$

with B_2 being the potential for the NS field strength H_3. The non-vanishing components of the RR fields in flat superspace are

$$G_{\underline{\alpha\beta cd}} = -i(\Gamma_{\underline{cd}})_{\underline{\alpha\beta}} \tag{5.16}$$

$$G_{\underline{\alpha\beta}} = -i(\Gamma_{11})_{\underline{\alpha\beta}} \,. \tag{5.17}$$

It is now straightforward to verify that

$$W_4 = dK_3 \,, \tag{5.18}$$

where all of the components of K_3 vanish except for K_{abc} which is given by

$$K_{abc} = \epsilon_{abc} \ K \ , \tag{5.19}$$

with

$$K = \frac{1-4y}{1+4y} \ . \tag{5.20}$$

The part of the GS Lagrangian arising from K is then given by

$$K_{mnp}| = (E_m{}^a E_n{}^b E_p{}^c)\epsilon_{abc} \ K| \ . \tag{5.21}$$

However, $E_m{}^a| = \mathcal{E}_m{}^a$ is again simply the dreibein for the induced GS metric,

$$\mathcal{E}_m{}^a\mathcal{E}_n{}^b\eta_{ab} = g_{mn} \ . \tag{5.22}$$

From this we derive

$$K_{mnp}| = \epsilon_{mnp}\sqrt{-\det g} \ \times K| \ . \tag{5.23}$$

It remains to show that K is proportional to det $\sqrt{\delta_m{}^n + \mathcal{F}_m{}^n}$. To this end, we first observe that we can replace the coordinate indices appearing in the Born-Infeld determinant as written here by orthonormal ones at no cost. Thus we can work with $\mathcal{F}_{ab}$ and h_{ab} and then evaluate at $\theta = 0$. We have

$$\begin{aligned} \det(1+\mathcal{F}) &= \exp \operatorname{tr} \log(1+\mathcal{F}) \\ &= \exp \operatorname{tr}\left(-\tfrac{\mathcal{F}^2}{2} + \tfrac{\mathcal{F}^4}{4} - \ldots\right) \ , \end{aligned} \tag{5.24}$$

where the second step follows from the antisymmetry of $\mathcal{F}$. Writing $\mathcal{F}$ in terms of h using (5.11) and employing the identity

$$h^3 = yh \; , \tag{5.25}$$

we find

$$\begin{aligned} \det(1+\mathcal{F}) &= \exp \operatorname{tr} h^2 \left(\tfrac{-4^2}{2(1+4y)^2} + \tfrac{4^4}{4(1+4y)^4} - \ldots\right) \\ &= \exp\log\left(1 - \tfrac{16y}{(1+4y)^2}\right) \\ &= \left(\tfrac{1-4y}{1+4y}\right)^2 \; . \end{aligned} \tag{5.26}$$

Therefore, K defined in (5.20) is given by

$$K = \sqrt{\det(\delta_m{}^n + \mathcal{F}_m{}^n)} \; , \tag{5.27}$$

and that the GS Lagrangian is obtained from the general formulae (3.12) and (3.16) to be

$$L^0 = dx^m \wedge dx^n \wedge dx^p \epsilon_{mnp} \; \mathcal{L} \; . \tag{5.28}$$

with

$$\mathcal{L} = \sqrt{-\det(g_{mn} + \mathcal{F}_{mn})} - f^* C_3 - f^* C_1 \mathcal{F} \; , \tag{5.29}$$

in agreement with the general results for Dp-brane actions in the GS formalism [27, 28, 29, 30, 31, 32].

6. The membrane in $N = 1,\; D = 4$ superspace

The final example we shall consider is the membrane in $N = 1,\; D = 4$ superspace. This is a type I brane for which the standard embedding condition defines an off-shell multiplet. Actually, this brane has codimension one and the worldsurface multiplet in question is an entire scalar superfield, but this is is simply the off-shell scalar multiplet in three dimensions.

To simplify the discussion we shall take the target space to be flat. For the embedding matrix we can take, as before,

$$E_\alpha{}^{\underline{a}} = 0 \, , \tag{6.1}$$

$$E_\alpha{}^{\underline{\alpha}} = u_\alpha{}^{\underline{\alpha}} + h_\alpha{}^{\beta'} u_{\beta'}{}^{\underline{\alpha}} \; , \tag{6.2}$$

where both α and α' are $d = 3$ spinor indices taking two values. We also choose

$$E_a{}^{\underline{a}} = u_a{}^{\underline{a}} \; . \tag{6.3}$$

The dimension zero component of the torsion equation (5.2) gives

$$h_\alpha{}^{\beta'} = i\delta_\alpha{}^{\beta'} h \tag{6.4}$$

and

$$T_{\alpha\beta}{}^{c} = -i(1+h^2)(\gamma^c)_{\alpha\beta} \ , \tag{6.5}$$

but in this case, the (real) field h is not related to a gauge field, rather its leading component is the auxiliary field in the scalar multiplet.

The Wess-Zumino form W_4 is in this case simply the pull-back of the target space four-form $G_4 = dC_3$ to M. The only non-vanishing component of G_4 for a flat target space is

$$G_{\underline{\alpha}\,\underline{\beta}\,\underline{c}\underline{d}} = -i(\Gamma_{\underline{cd}})_{\underline{\alpha}\,\underline{\beta}} \ . \tag{6.6}$$

The general argument given previously implies that

$$W_4 = dK_3 \ . \tag{6.7}$$

It is straightforward to verify that this is indeed the case, and that the only non-vanishing component of K_3 is

$$K_{abc} = \epsilon_{abc}\ K \ , \tag{6.8}$$

where

$$K = \frac{1-h^2}{1+h^2} \ . \tag{6.9}$$

The GS Lagrangian form is

$$L_{mnp} = (K_{mnp} - (f^*C)_{mnp})| \ , \tag{6.10}$$

from which the GS Lagrangian density is found to be

$$\mathcal{L} = \sqrt{-\det g}\ \left(\frac{1-h^2}{1+h^2}\right)| - \frac{1}{6}\epsilon^{mnp}(f^*C)_{mnp}| \ , \tag{6.11}$$

where g is again the standard GS induced metric. The only difference from the usual GS Lagrangian is the factor multiplying the GS measure containing as it does the auxiliary field h. However, the equation of motion for this field is purely algebraic and can be used to set $h = 0$. We thus recover the standard GS action. It is amusing to note that the off-shell action given here has the same form as the DBI action for the D2-brane when expressed in terms of h.

Since the multiplet is off-shell it is possible to construct a superspace action for this model using the techniques that were introduced in [7] in the context of the heterotic string. In order to do this it is useful to introduce the notion of a q-vector density, which we shall call a q-coform for short. Such an object is a tensor density of tensorial type (q,0) which is totally antisymmetric. There is a natural pairing between q-coforms P and q-forms ω given by

$$(P,\omega) = \int P^{M_q\ldots M_1}\omega_{M_1\ldots M_q} \ . \tag{6.12}$$

If the space of q-coforms is denoted by $\tilde{\Omega}_q$, then there is a natural derivative $\tilde{d}$ which maps $\tilde{\Omega}_q$ to $\tilde{\Omega}_{q-1}$ and which satisfies $\tilde{d}^2 = 0$. In a coordinate basis one has

$$(\tilde{d}P)^{M_1 \dots M_{q-1}} = P^{M_1 \dots M_q}{}_{,M_q} := P^{M_1 \dots M_q} \overleftarrow{\partial}_{M_q} \ , \tag{6.13}$$

where the derivative is the right derivative. One then has

$$(P, d\omega) = -(\tilde{d}P, \omega) \ , \tag{6.14}$$

up to possible surface terms. Using this notation we can write an action for the membrane in the form

$$S = (P, K - f^*C - dQ) \ , \tag{6.15}$$

where P is a three-coform and Q a new two-form field. This action is invariant under

$$P \to P + \tilde{d}X \ , \tag{6.16}$$

where X is a four-coform. Varying the action with respect to Q gives

$$\tilde{d}P = 0 \ . \tag{6.17}$$

Thus P is an element of the third homology group associated with the operator $\tilde{d}$. This group is one-dimensional, but non-trivial, and can be represented in suitable coordinates by

$$P^{mnp} = \theta^2 \epsilon^{mnp} \times \text{constant} \tag{6.18}$$

with all other components vanishing. The constant of integration is naturally interpreted as the tension of the brane. Varying the action with respect to the embedding is quite complicated, but because we know the content of the worldsurface multiplet we can instead substitute (6.18) back into the action to get the Green-Schwarz form given above.

In constructing this action we have assumed that the basic embedding condition (2.2) is satisfied. However, one can also derive this by including a Lagrange multiplier field $\Pi_{\underline{a}}{}^{\alpha}$ to impose the embedding constraint. The action then becomes

$$S = \int \Pi_{\underline{a}}{}^{\alpha} E_{\alpha}{}^{\underline{a}} - (P, K - f^*C - dQ) \ . \tag{6.19}$$

This action has precisely the same structure as the action given for the heterotic string. Although such actions have been written down previously for p-branes with $p \geq 2$, it was not known at the time whether such actions would lead to the corerct brane dynamics. If the embedding condition imposed by the Lagrange multiplier superfield leads to an on-shell world-volume multiplet then more degrees of freedom, contained within the Lagrange multiplier superfield, will also propagate. The example studied here is an off-shell multiplet and so would seem to provide the first well-established example of a superfield action for a brane with $p > 1$.

We conclude this section with two comments. Firstly, the form of the action integral given in (6.15) can also be used when the constraints are on-shell. In such a case, the resulting action should not be thought of as a superfield action, but simply as the GS action rewritten in superspace. Secondly, in order to write down the action for the $D=4$ membrane in the form given in (6.19), the embedding constraint must be relaxed. However, even in this case there will still be a globally defined three-form K such that $dK_3 = W_4$, although the explicit form for K will be more complicated than it is when the embedding condition is satisfied.

7. Comments

In this paper we have shown that one can construct Green-Schwarz actions for almost all branes, excluding those with self-dual gauge fields, in a systematic fashion starting from the Wess-Zumino form on the worldvolume. It is important that one uses the superembedding formalism to derive this result because the Wess-Zumino form vanishes identically on the bosonic worldvolume. The resulting superspace Lagrangian form, L_{p+1}, given in (3.12) can be obtained explicitly if one know the $(p+1)$-form K_{p+1} for a given brane. In fact, if the standard embedding condition (2.2) holds, it is straightforward to invert the relation

$$W_{p+2} = dK_{p+1}\,. \tag{7.1}$$

This is because the only non-vanishing component of K in this case will be the one which has only vectorial indices while the only non-vanishing component of W will be the one with two spinor indices and p vectorial indices, provided that the background geometry is of standard type (which would be expected to arise form brane integrability in any case). In this situation one would have

$$W_{\alpha\beta c_1\ldots c_p} = T_{\alpha\beta}{}^{c_o} K_{c_o c_1\ldots c_p}\,. \tag{7.2}$$

From this equation one determines K to be

$$K_{a_1\ldots a_{p+1}} = t_{a_1}{}^{\alpha\beta} W_{\alpha\beta a_2\ldots a_{p+1}}\,, \tag{7.3}$$

where $t_a{}^{\alpha\beta}$ is the inverse of $T_{\alpha\beta}{}^c$,

$$t_a{}^{\alpha\beta} T_{\alpha\beta}{}^b = \delta_a{}^b. \tag{7.4}$$

Note that the right-hand side of this equation is totally antisymmetric on the vector indices although this is not manifest.

This explicit form for K was found previously in the case of Fp-branes in [7, 9, 10] where the Lagrangian form was referred to as $\tilde{B}$. For these branes the dimension zero worldvolume torsion and the inverse tensor t have components proportional to the components of the Dirac gamma matrices. In all these case one then finds

$$K_{a_1 \ldots a_p} = \epsilon_{a_1 \ldots a_p} \tag{7.5}$$

and this in turn results in the standard Nambu kinetic term in the GS action proportional to the bosonic worldvolume in the induced metric.

In the general case $T_{\alpha\beta}{}^c$ involves the worldvolume field h and so the expression for K will be more complicated. It will be equal to the epsilon tensor times a scalar factor which, for example in the case of D-branes, will be the Born-Infeld function as we saw explicitly in the case of the D2-brane. However, it is worth emphasizing that in our formulation no knowledge of the DBI action is assumed and it is derived from the first principles described in the paper. Consequently our formalism can be applied to construct new brane actions which will involve generalisations of DBI actions with higher rank field strengths. The formulation of [14], applied to Dp-branes, in essence provides the form K_{p+1} as the Dirac-Born-Infeld (DBI) kinetic term. However, it should be emphasized that the knowledge of the usual GS type formulation of the Dp-brane action is used as an input in this construction, thereby essentially elevating the known DBI action to an integral in a bosonic slice of the worldvolume supermanifold.

We have also shown that if the embedding condition leads to an off-shell multiplet one can construct superspace actions using the ideas introduced in [7] and illustrated this with the example of the membrane in $D = 4$. It would be interesting to determine the gauge fixed action for this membrane in terms of worldvolume superfields in the static gauge. This should give a manifestly supersymmetric superspace action expressed in terms of physical superfields. One would hope to recover in this way a manifestly supersymmetric, superfield formulation of the results obtained some time ago in [33] by gauge fixing the Green-Schwarz action accompanied by a complicated set of field redefinitions.

The calculation of K_7 for the M5-brane would also of considerable interest, because a self-dual worldvolume 3-form field strength is involved. Manifestly Lorentz invariant actions for such forms do not exist unless one introduces an auxiliary scalar field and new gauge invariances, fixing of which necessarily breaks Lorentz invariance [20, 21]. A generalized action principle fails due to the very presence of this auxiliary field [13]. On the other hand, a generalized action principle for self-dual supergravity in six-dimensions is known to exist [34]. The price one pays is that the action is not supersymmetric when restricted to x-space. Instead, one should vary the action first in the full group manifold, and then restrict the result to x-space and this leads to supersymmetric and consistent equations of motion. Determining K_7 for the M5-brane would shed light on the question of whether a similar phenomenon could occur in the M5-brane. In this context, it is worth mentioning the work of [35] where κ-symmetric M5-equations of motion are obtained from an action which is not κ-invariant as it contains three-forms of both dualities.

JHEP08(1998)011

To conclude, we emphasize that the action formula proposed in this paper is a universal one which applies to all branes with the possible exception of those which contain chiral forms. In addition to producing the known brane actions in the GS formalism (upon restriction to bosonic wolrdvolume), our action fomula solves the problem of constructing actions for branes whose worldvolume supports supermultiplets that contain higher than second rank antisymetric tensors. Indeed, we shall apply the action formula of this paper to construct an action for an L5-brane in $D = 9$ which contains a linear multiplet with a four-form potential [25]. Other applications of the action formula might yield new insights into duality symmetries that map branes into other branes, as well as facilitating the study of interesting residual symmetries such as superconformal symmetry in various backgrounds of interest.

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